DISSERTATIO ACADEMICA

THEORIAM ÆQUATIONUM FUNCTIONALIUM DUARUM VARIABILIUM EJUSQUE IN DOCTRINA SERIERUM USUM EXHIBENS;

QUAM

CONSENSU AMPLISS. FACULTATIS PHILOSOPH.

AD IMPERIALEM ACAD. ABOËNSEM,

PRÆSIDE

Mag. NATH. G. AF SCHULTÉN,

Mathematatum Professore Publ. & Ord.,
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PRO GRADU PHILOSOPHICO

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horis p. m. solitis.

P. II.

ABOÆ, Typis FRENCKELLIANIS.

11.

signumque f_1 in f mutando, æquatio igitur in sequentibus consideranda fiet

$$F'' (x, y_x, y_{fx}, y_{f^2x}, y_{f^3x}, \ldots . y_{f^n x}) = 0 \; . \; 3),$$

datis quomodocumque formis functionum F'' et f.

§. IV.

Ad simpliciorem longe formam revocari hanc posse æquationem ante omnia notatu dignum est. Data scilicet quomodocumque forma functionis f, reduci semper ipsa potest 3) ad alias duas, ubi fx valorem habeat simplicissimum

$$x + 1;$$

quod sequenti quidem fiet modo:

Positis

$$x = v_z, \; fx = v_{z+1},$$

ubi v functio est incognita variabilis novæ z, habebitur

$$v_{z+1} = fv_z \; . \; . \; . \; . \; 4).$$

Quæ quidem æquatio, cum pro omnibus ipsius z valoribus obtinere censenda sit, fiet porro

 v_{z+2}

$$v_{z+2} = fv_{z+1} = ffx = f^2x$$

$$v_{z+3} = fv_{z+2} = fffx = f^3x$$

$$v_{z+4} = fv_{z+3} = ffffx = f^4x$$

$$\cdots\cdots\cdots\cdots$$

$$v_{z+n} = \cdots\cdots = f^nx.$$

Ponatur ulterius

$$y_{v_z} = w_z,$$

designante w iterum functione incognita, erunt utique

$$y_{v_{z+1}} = w_{z+1}$$

$$y_{v_{z+2}} = w_{z+2}$$

$$\cdots\cdots$$

$$y_{v_{z+n}} = w_{z+n};$$

hincque igitur

$$y_x = w_z$$

y_{fx}

$$y_{fx} = w_{z+1}$$

$$y_{f^2x} = w_{z+2}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$y_{f^nx} = w_{z+n}:$$

quibus quidem valoribus una cum valore ipsius x in ipsa 3) substitutis, abibit eadem in

$$F'' \left(v_z, w_z, w_{z+1}, w_{z+2}, \ldots . w_{z+n}\right) = 0 \ldots 5).$$

Determinationem generalem functionum incognitarum v atque w revocatam jam videmus ad resolutionem æquationum 4) et 5), quarum utramque forma universali

$$F \left(x, y_x, y_{x+1}, y_{x+2}, \ldots . y_{x+n}\right) = 0 \ldots . 6)$$

contineri manifestum est, quæ ea omnino est, in quam ipsa abit 3), posito

$$fx = x + 1.$$

Datis vero formis ipsarum v et w, per eliminationem ipsius z æquationes inter

$$x = v_z,\ y_x = w_z,$$

nec non investigationem in æquatione hinc resultante ipsius y_x, determinatum iri formam functionis quæsitæ y_x, æquationi de qua agitur 3) satisfacientis, in aprico est.

§. V.

Allatæ nuper æquationis 6), quam in sequentibus igitur tantum considerabimus, tractationem adgressuri, eo quidem, ut idoneo maxime, progrediemur ordine, ut primum ipsa tantummodo resultata, ad quæ hac in materia pervenerint Mathematici, universalissima succincte exponamus, quo sic scilicet in tabulam quasi redacta ad applicationes quascumque instituendas in promtu optime habeantur, dein vero, propositis scilicet æquationem de qua agitur resolvendi regulis fundamentalibus, demonstrationem earumdem breviter persequamur, deinque ulterius exemplis quibusdam particularibus easdem illustremus regulas, tandemque earumdem in doctrina serierum usum exhibeamus.

Resultatorum igitur ipsorum expositionem primum quod attinet, in duas commodissime partes dispesci eamdem observamus, priorem scilicet, qua æquatio data 6) primi tantum habeatur respe-

ctu

ctu functionis quæsitæ y_x gradûs, posterioremque qui ceteros omnes casus complectatur.

Ut illam igitur, quæ præcipuas, quas adferendas habemus, regulas comprehendit, adgrediamur, æquationem nostram 6), quæ respectu y_x primi tantum accipienda jam est gradus, forma jam in genere venire sequenti notandum est:

$$y_{x+n} + py_{x+n-1} + qy_{x+n-2} + \ldots ty_x + u = o \ldots 7),$$

ubi p, q, . . t, u functiones sunt quæcumque datæ variabilis x.

Casus hujusce æquationis simplicissimus, cujus theoria tractatio ceterorum fere omnium nititur, is est, ubi $n = 1$, $p = o$, $q = o$, . . $t = -1$, $u = -v$; unde formam igitur induet

$$y_{x+1} - y_x - v = o \ldots 8).$$

Quandoquidem consideratio hujusce æquationis simplicis præcipui est hoc in argumento momenti, signo particulari uti solent Mathematici ad rationem definiendam qua e functione data v pendere censenda sit incognita y_x. Ponunt scilicet

$$y_x = \Sigma v.$$

Hunc

Hunc nos etiam secuturi morem regulas igitur ante omnia jam exponemus generalissimas, quas eruere valuerint Analystæ ad investigandam formam functionis Σv ex data illa v, quas autem arctissimis omnino circumscriptas haberi limitibus, ab initio statim observasse conveniet.

Notandæ hunc in finem sunt formulæ sequentes fundamentales:

1:o Designantibus p, q, r, &c. functionibus quibuscumque ipsius x, k vero quantitate sive ab x non pendente, sive etiam functione quacumque ipsius x ejus indolis, ut non mutetur abeunte x in $x + 1$, qualis est v. gr.

$$\varphi\,(\text{Sin}\ 2\pi x,\ \text{Cos}\ 2\pi x),$$

ubi φ functio est arbitraria et $\pi = 3{,}14159\ldots$, habebuntur utique relationes

$$\Sigma\,(p \pm q \pm r \pm \&c.) = \Sigma p \pm \Sigma q \pm \Sigma r \pm \&c.$$

$$\Sigma kp = k\Sigma p$$

$$\Sigma k = kx$$

$$\Sigma o = k,$$

observandumque quoque est, si inventum fuerit

$$\Sigma p = \psi x,$$

haberi

haberi in genere

$$\Sigma p = k + \psi x:$$

ex qua ultima relatione cernitur signum Σ non tantum eadem gaudere proprietate ac in calculo Integrali signum $\int$, ut adjiciendæ scilicet ansam præbeat constanti arbitrariæ cuicumque k, sed indeterminatiora etiam adhuc longe haberi resultata ex signo Σ pendentia ac ea quæ signi sint $\int$, cum non tantum quantitas simpliciter constans, verum etiam functio ipsius variabilis x, allatæ nuper indolis, censenda isthæc sit k.

2:o Obtinebit ulterius, quæcumque fuerit ipsa v functio, formula

$$\Sigma v = \int v dx - \tfrac{1}{2} v + \frac{b_1}{1 \cdot 2} \cdot \frac{dv}{dx} - \frac{b_3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{d^3 v}{dx^3}$$

$$+ \frac{b_5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{d^5 v}{dx^5} - \&c. \quad . \quad . \quad . \quad . \quad a)$$

denotantibus b_1, b_3, b_5, &c. primo, secundo, tertio, &c. numerorum Mathematicis notissimorum (*Bernoullianorum* olim dictorum nomine), quorum sequens habetur lex per se manifesta:

$$b_1 = \frac{1}{6}$$

$$b_3 = \frac{4 \cdot 3}{1 \cdot 2} \cdot \tfrac{1}{5}\, b_1^{\,2}$$

b_5

$$b_5 = \frac{6.5}{1.2} \cdot \tfrac{2}{7}\, b_1 \cdot b_3$$

$$b_7 = \frac{8.7}{1.2} \cdot \tfrac{2}{9}\, b_1 \cdot b_5 + \frac{8.7.6.5}{1.2.3.4} \cdot \tfrac{1}{9} \cdot b_3^2$$

$$b_9 = \frac{10.9}{1.2} \cdot \tfrac{2}{11} \cdot b_1 \cdot b_7 + \frac{10.9.8.7}{1.2.3.4} \cdot \tfrac{2}{11} \cdot b_3 \cdot b_5$$

$$b_{11} = \frac{12.11}{1.2} \cdot \tfrac{2}{13} \cdot b_1 \cdot b_9 + \frac{12.11.10.9}{1.2.3.4} \cdot \tfrac{2}{13} \cdot b_3 \cdot b_7$$
$$+ \frac{12.11.10.9.8.7}{1.2.3.4.5.6} \cdot \tfrac{1}{13} \cdot b_5^2$$

$$b_{13} = \frac{14.13}{1.2} \cdot \tfrac{2}{15} \cdot b_1 \cdot b_{11} + \frac{14.13.12.11}{1.2.3.4} \cdot \tfrac{2}{15} \cdot b_3 \cdot b_9$$
$$+ \frac{14.13.12.11.10.9}{1.2.3.4.5.6} \cdot \tfrac{2}{15} \cdot b_5 \cdot b_7$$

$$b_{15} = \frac{16.15}{1.2} \cdot \tfrac{2}{17} \cdot b_1 \cdot b_{13} + \frac{16.15.14.13}{1.2.3.4} \cdot \tfrac{2}{17} \cdot b_3 \cdot b_{11}$$
$$+ \frac{16.15.14.13.12.11}{1.2.3.4.5.6} \cdot \tfrac{2}{17} \cdot b_5 \cdot b_9 + \frac{16.15.14.13.12.11.10.9}{1.2.3.4.5.6.7.8} \cdot \tfrac{1}{17} \cdot b_7^2$$

$$b_{17} = \frac{18.17}{1.2} \cdot \tfrac{2}{19} \cdot b_1 \cdot b_{15} + \frac{18.17.16.15}{1.2.3.4} \cdot \tfrac{2}{19} \cdot b_3 \cdot b_{13}$$
$$+ \frac{18.17.16.15.14.13}{1.2.3.4.5.6} \cdot \tfrac{2}{19} \cdot b_5 \cdot b_{11} + \frac{18.17.16.15.14.13.12.11}{1.2.3.4.5.6.7.8} \cdot \tfrac{2}{19} \cdot b_7 \cdot b_9$$

&c. &c.

quorumque, quia magni sunt hoc in argumento usus, quindecim attulisse juvabit primos, qui sequentes sunt: